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OPTIMAL CONVERGENCE OF MINIMUM NORM APPROXIMATIONS IN HP.(U)

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Optimal Convergence of Minimum Norm Approximations in H_p.

by

Frank Stenger

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Abstract

Let $1 < p \leq \infty$, and let $H_p(U)$ denote the family of all functions f that are analytic in the unit disc U such that

$$||f||_p = \lim_{r \to 1^-} (\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty .$$

Set

$$\sigma_n = \inf_{w_j, x_j} \sup_{f \in H^p(U), ||f||_p = 1} \left| \int_{-1}^{1} f(x)dx - \sum_{j=1}^{n} w_j f(x_j) \right| .$$

It is shown that given any $\varepsilon > 0$, there exists an integer $n(\varepsilon) \geq 0$, such that if $n > n(\varepsilon)$ and $q = p/(p - 1)$, then

$$\exp\{-(5^{1/2}\pi + \varepsilon)n^{1/2}\} \leq \sigma_n \leq \exp\{-(\frac{\pi}{(2q)^{1/2}} - \varepsilon)n^{1/2}\} ,$$

Let $H_p^*(U)$ denote the family of all functions f such that $g \in H_p(U)$, where $g(z) = f(z)/(1 - z^2)$, and where $H_p^*(U)$ is normed by $||f||_p^* = ||g||_p$, $||g||_p$ being defined as above. Let $\{T_n(f)\}_{n=1}^{\infty}$ be an approximation scheme defined by

$$T_n(f)(z) = \sum_{j=1}^{n} f(x_j)\phi_{n,j}(z), \qquad f \in H_p^*(U) ,$$

where $\phi_{n,j}$ is analytic in U, and such that $||T_n(f)||_p^* \leq C||f||_p^*$, where $C > 1$, but independent of n. Then given any $\varepsilon > 0$, there exists an integer $n(\varepsilon) \geq 0$, such that whenever $n > n(\varepsilon)$, then

$$\exp\{-(5^{1/2}\pi + \varepsilon)n^{1/2}\} \leq \inf_{T_n} \sup_{f \in H_p^*(U), ||f||_p^* = 1} \sup_{-1<x<1} |f(x) - T_n(f)(x)|$$

$$\leq \exp\{-(\frac{\pi}{2q^{1/2}} - \varepsilon)n^{1/2}\} .$$

1. Introduction and Summary

Let $1 < p \leq \infty$, and let $H_p(U)$ denote the family of all functions f that are analytic in the unit disc U of the complex plane, such that

$$(1.1) \qquad \|f\|_p = \lim_{r\to 1^-}\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} < \infty$$

Let σ_n be defined by

$$(1.2) \qquad \sigma_n = \inf_{w_j\in\mathbb{C},\, x_j\in U,} \left\{ \sup_{f\in H^p(U),\, \|f\|_p = 1} \left| \int_{-1}^{1} f(x)dx - \sum_{j=1}^{n} w_j f(x_j) \right| \right\}.$$

and q by $q = p/p-1$.

We then prove

Theorem 1.1: Given any $\varepsilon > 0$ there exists an integer $n(\varepsilon) \geq 0$ such that whenever $n > n(\varepsilon)$, then

$$(1.3) \qquad \exp\{-(5^{\frac{1}{2}}\pi + \varepsilon)n^{\frac{1}{2}}\} \leq \sigma_n \leq \exp\{-(\frac{\pi}{(2q)^{\frac{1}{2}}} - \varepsilon)n^{\frac{1}{2}}\} \quad .$$

Next, let $H_p^*(U)$ denote the family of all functions f such that $g \in H_p(U)$, where $g(z) = f(z)/(1 - z^2)$, and normed by $\|f\|_p^* = \|g\|_p$, where $\|g\|_p$ is defined as in (1.1). Let $\{T_n(f)\}_{n=1}^{\infty}$ be a linear approximation scheme defined by

$$(1.3) \qquad T_n(f)(z) = \sum_{j=1}^{n} f(x_j)\phi_{n,j}(z)$$

where $\phi_{n,j}$ is analytic in U for each n and j, and such that

$$(1.4) \qquad \|T_n(f)\|_p^* \leq C\|f\|_p^*$$

where C is independent of n . We then prove

<u>Theorem 1.2</u>: Given any $\varepsilon > 0$, there exists an integer $n(\varepsilon) \geq 0$ such that whenever $n > n(\varepsilon)$,

$$\exp\{-(5^{1/2}\pi + \varepsilon)n^{1/2}\}$$

$$(1.5) \qquad \leq \inf_{T_n} \sup_{f \in H_p^*(U), ||f||_p^* = 1} \sup_{-1<x<1} \left| f(x) - T_n(f)(x) \right|$$

$$\leq \exp\{-(\frac{\pi}{2q^{1/2}} - \varepsilon)n^{1/2}\}$$

We remark that the condition (1.4) ensures that $T_n(f) \to f$ for all $f \in H_p^*(U)$.

Let us briefly mention some other papers which are relevant to the present work. In 1964 Wilf [23] proved for the case $p = 2$ that $\sigma_n = O[(\log n/n)^{1/2}]$. In 1971 Haber [6] and Johnson and Riesz [8] proved (for $p = 2$) that $\sigma_n^2 = O(1/n)$. In 1973 [17] it was shown by the author that for the case $p = 2$, $\sigma_n = O(e^{-\pi/2\, n^{1/2}})$. In 1975 it was shown by Loeb and Werner [9] that for arbitrary $p > 1$, $\sigma_n \leq 2^{1+2/q} \exp[-(n/2)^{1/2}/(2q)]$, where $1/p + 1/q = 1$.

The bounds obtained in the present paper are sharper than any others that have been obtained previously. Moreover, while there is a gap in our upper and lower bounds, no one has previously obtained a lower bound. Finally, no one has previously obtained any bounds of the type in Thm. 1.2, for H_p^*-types interpolation.

It is convenient for purposes of the discussion which follows, to set

$$(1.6) \qquad a = \left(\frac{\pi d}{q}\right)^{1/2}, \quad b = 5^{1/2}\pi$$

and to assume, throughout, that ε is an arbitrary positive number.

The results of Thms 1.1 and 1.2 may be easily extended to establishing the optimal $O(e^{-cn^{1/2}})$ rate of convergence in other H_p spaces, $p > 1$, and to deduce lower bounds on the rates of convergence of other methods of approximations. In what follows, we shall describe some of these. We shall also mention known methods of approximation in each case, which converge at the $O(e^{-\gamma n^{1/2}})$ rate. At this time it is not known whether or not $c = \gamma$ for any of these methods.

(a) Let $0 < d \leq \pi/2$, let $\mathcal{D}_d = \{z=x+iy: |\arg[(1+z)/(1-z)]| < d\}$ (Note that $\mathcal{D}_{\pi/2} = U$) and let $H_p(\mathcal{D}_d)$ denote the family of all functions f that are analytic in $\mathcal{D}_d$ such that

$$(1.7) \qquad ||f||_p = \liminf_{C\to\partial\mathcal{D}_d,\ C\subset\mathcal{D}_d} \left(\int_C |f(z)|^p\,|dz|\right)^{1/p} < \infty.$$

The optimal rate of convergence of quadratures (1.2) in $H_p(\mathcal{D}_d)$ is $O(e^{-cn^{1/2}})$ where

$$(1.8) \qquad a \leq c \leq b + \varepsilon$$

The quadrature methods of Thm. 1.6 (b) of [17] and Thm. 3.2 of [18] converge at the $O(e^{-an^{1/2}})$ rate. The method constructed via the proof in [9] converges at the $O(\exp\{-[d^{1/2}/(2^{3/2}q)]n^{1/2}\})$ rate.

(b) Let $H_p^*(\mathcal{D}_d)$ denote the family of all functions g such that $f \in H_p(\mathcal{D}_d)$, where $f(z) = g(z)/(1-z^2)$ ane where $H_p(\mathcal{D}_d)$ is defined in (a) above. The optimal rate of convergence of interpolation (1.3) in $H_p^*(\mathcal{D}_d)$ is $O(e^{-cn^{1/2}})$, where

$$(1.9)\qquad 2^{-1/2}a - \varepsilon \leqq c \leqq b + \varepsilon .$$

The method [19]

$$f(x) \approx \sum_{j=-N}^{N} f(x_j) S(j,h) \circ \log\left(\frac{1+x}{1-x}\right)$$

$$(1.10)\qquad S(j,h)(x) = \frac{\sin[\pi(x-jh)/h]}{\pi(x-jh)/h}$$

$$h = \left(\frac{\pi d q}{N}\right)^{1/2}, \quad x_j = \tanh(jh/2)$$

converges at the $O(\exp\{-[2^{-1/2}a-\varepsilon]n^{1/2}]$ rate, where $n = 2N+1$.

(c) Let $k \geqq 0$ be an integer, and let $H_p^k(\mathcal{D}_d)$ denote the family of *all functions* f *such that* $g_k \in H_p(\mathcal{D}_d)$ *where* $H_p(\mathcal{D}_d)$ is defined as in (a) above, and where

$$(1.11)\qquad g_k(z) = \frac{k!}{2}\left[\frac{(-1)^k}{(1+z)^{k+1}} + \frac{1}{(1-z)^{k+1}}\right] f(z)$$

Let $T_n(f)$ be defined as in (1.3), and set

$$(1.12)\quad \varepsilon_n^k = \sup_{g_k \in H_p(\mathcal{D}_d), \|g_u\|_p = 1} \left\{ \max_{j=0,1,\ldots,k} \sup_{x \in (-1,1)} |f^{(j)}(x) - \{T_n(f)\}^{(j)}(x)| \right\}$$

Then the optimal rate of convergence to zero of ε_n^k is $O(e^{-cn^{1/2}})$, where c is subject to (1.9). If $T_n(f)$ is defined by

$$(1.13)\qquad T_n(f)(x) = \sum_{j=-N}^{N} \frac{f(x_j)}{(1-x_j^2)^k} (1-x^2)^k S(j,h) \circ \log\left(\frac{1+x}{1-x}\right) \quad \text{where}$$

where $n = 2N + 1$, and where x_j , $S(j,h)$ and h are defined as in (1.10), then [10]

$$(1.14)\qquad \max_{j=0,1,\ldots,k}\ \sup_{x\in(-1,1)} \left|f^{(j)}(x) - \{T_n(f)\}^{(j)}(x)\right|$$

$$= O(\exp\{-(2^{-\frac{1}{2}}a-\varepsilon)n^{\frac{1}{2}}\})\,||g_k||_p$$

(d) Let $\mathcal{D}_d = \{z = x+iy: |y|<d\}$, and let $H_p(\mathcal{D}_d)$ denote the family of all functions f that are analytic in $\mathcal{D}_d$, such that

$$(1.15)\qquad N(f,y) = \left(\int_R \{|f(x+iy)|^p + |f(x-iy)|^p\} \cosh^{2p/q}(x/2)dx\right)^{1/p} < \infty$$

$(R = (-\infty,\infty))$ for all y , $0 \leq y < d$, and such that $||f||_p = N(f,d^-) < \infty$.

(i) The optimal rate of convergence of n-point quadratures $\int_R f(x)dx \simeq \sum_{j=1}^{n} w_j\, f(x_j)$ in $H_p(\mathcal{D}_d)$ is $O(e^{-cn^{\frac{1}{2}}})$, where c is subject to (1.8). The trapezoidal rule, $\int_R f(x)dx \simeq h \sum_{j=-N}^{N} f(jh)$, $h = (2\pi dq/N)^{\frac{1}{2}}$, converges at the $O(e^{-an^{\frac{1}{2}}})$ rate, where $n = 2N+1$.

(ii) The optimal rate of convergence of interpolation of $f \in H_p(\mathcal{D}_d)$ over R is $O(e^{-cn^{\frac{1}{2}}})$, where c is subject to (1.9). Interpolation via the Whittaker Cardinal function, (see Eqs. (1.10))

$$f(x) \simeq \sum_{j=-N}^{N} f(jh)\, S(j,h)(x)\ ,\ h = (\pi dq/N)^{\frac{1}{2}}\ ,$$

converges at the $O(\exp\{-(2^{-\frac{1}{2}}a-\varepsilon)n^{\frac{1}{2}}\}$ rate [19].

(iii) The optimal rate of convergence of the approximation

$$\int_R e^{ixt} f(x)dx \approx \sum_{j=1}^{n} w_j f(x_j) e^{ix_j t} \text{ , for } f \in H_p(\mathcal{D}_d) \text{ and } |t| < [n\pi/(2qd)]^{\frac{1}{2}}$$

is $O(e^{-cn^{\frac{1}{2}}})$, where $2^{-\frac{1}{2}} a \leq c \leq b$. The method $\int_R f(x) e^{ixt} dx \approx$

$$h \sum_{j=-N}^{N} f(jh)\, e^{ijht} \text{ , } h = (\pi dq/N)^{\frac{1}{2}} \text{ (The Fast Fourier transform method)}$$

converges at the $O(e^{-2^{-\frac{1}{2}} an^{\frac{1}{2}}})$ rate [19].

(iv) The optimal rate of convergence to zero of the error

$$(1.16) \qquad \sup_{t \in R} \left| \text{ P.V.} \int_R \frac{f(x)}{x-t}\, dx - \sum_{j=1}^{n} f(x_j)\, \phi_{n,j}(t) \right|$$

where P.V. denotes the principal value and $f \in H_p(\mathcal{D}_d)$ is $O(e^{-cn^{\frac{1}{2}}})$, where c is subject to (1.9). The method [19]

$$(1.17) \qquad \text{P.V.} \int_R \frac{f(x)}{x-t}\, dx = \sum_{j=-N}^{N} f(jh)\left(\frac{x-jh}{2h}\right)\left\{S(0,1)\circ\left(\frac{x-jh}{2h}\right)\right\}^2 ,$$

$h = (\pi dq/N)^{\frac{1}{2}}$, converges at the $O(\exp\{-(2^{-\frac{1}{2}}a-\varepsilon)n^{\frac{1}{2}}\})$ rate , where $n = 2N + 1$.

(e) Let $\mathcal{D}_d = \{z=x+iy: |\arg z| < d\}$ and let $H_p(\mathcal{D}_d)$ denote the family of all functions f that are analytic in $\mathcal{D}_d$ such that

$$(1.18) \qquad M(f,p,\theta) = \int_0^\infty (1+t)^{2p/q}\, |f(t\, e^{i\theta})|^p\, dt < \infty$$

for all $|\theta| < d$, and such that $\|f\|_p = \{M(f,p,-d^-) + M(f,p,d^-)\}^{1/p} < \infty$. The optimal rate of convergence of n-point quadratures $\int_0^\infty f(x)dx \approx \sum_{j=1}^{n} w_j f(x_j)$ in $H_p(\mathcal{D}_d)$ is $O(e^{-cn^{\frac{1}{2}}})$ where c is subject to

(1.8). The method

$$(1.19) \qquad \int_0^\infty f(x)dx \cong h \sum_{j=-N}^{N} e^{jh} f(e^{jh})$$

converge at the $O(e^{-an^{1/2}})$ rate [19], where $h = (2\pi dq/N)^{1/2}$ and $n = 2N+1$.

(f) Let $\mathcal{D}_d$ be defined as in (e) above and let $H_p(\mathcal{D}_d)$ denote the family of all functions f that are analytic in $\mathcal{D}_d$, such that

$$(1.20) \qquad N(f,p,\theta) = \int_0^\infty t^{-1-p/q} (1+t)^{2p/q} |f(fe^{i\theta})|^p d\theta < \infty$$

for all $|\theta| < d$, and such that $\|f\|_p = \{N(f,p,-d^-) + N(f,p,d^-)\}^{1/p} < \infty$. The optimal rate of interpolation $f(x) \cong \sum_{j=1}^{n} f(x_j)\varphi_{n,j}(x)$ over $(0,\infty)$ for f in $H_p(\mathcal{D}_d)$ converges at the $O(e^{-cn^{1/2}})$ rate, where c is subject to (1.9). The method

$$(1.21) \qquad f(x) \cong \sum_{j=-N}^{N} f(e^{jh})S(j,h)\circ \log x$$

converges at the $O(e^{-(2^{-1/2}a-\varepsilon)n^{1/2}})$ rate [19], where $h = (\pi d8/N)^{1/2}$ and $n = 2N + 1$.

(g) Consider the solution of the problem

$$(1.22) \qquad f(x) = \int_0^\infty k(x-t)f(t)dt+g(x) \ , \ x > 0$$

for f, given k and g, and where $\hat{k}$ and $\hat{g}$, the Fourier transforms of k and g, are known functions in $H_p(\mathcal{D}_d)$, as defined in (d) above. Then the optimal method of approximating f in the form (1.3) for arbitrary

k and g in $H_p(\mathcal{D}_d)$ converges at the rate $O(e^{-cn^{1/2}})$, where $2^{-1/2}a \leq c \leq b$. The method in [21] converges at the $O(e^{-2^{-1/2}an^{1/2}})$ rate.

(h) Consider the approximate solution of the Hilbert problem [7]. Let $\mathcal{D}_d$ be defined as in (a) above, let each of the m contours of the Hilbert problem be expressed by $L_j = \{\phi_j(t): 1 \leq t \leq 1\}$ where each ϕ_j is in $H_p(\mathcal{D}_d)$, and let the "data" G on each L_j be expressible in the form $g_j(t) = G(\phi_j(t))$, where $g_j \in H_p(\mathcal{D}_d)$ for each j. Then the optimal rate of convergence of an approximate solution (1.3) to the exact solution of the Hilbert problem is $O(e^{-cn^{1/2}})$, where c is subject to (1.8). The method in [7] converges at the $O(e^{-an^{1/2}})$ rate.

(j) Consider the solution of the Dirichlet problem $\Delta f = \lambda^2 f$ on the exterior of a bounded simply-connected domain B, whose boundary consists of a finite number of m contours L_j on which we are given boundary data G, as described in (h) above. Then the optimal rate of convergence of an approximate solution (1.3) to the exact solution f is $O(e^{-cn^{1/2}})$, where c is subject to (1.8). The method of solution in [3] converges at the $O(e^{-an^{1/2}})$ rate.

(k) Let $0 \in H_p^*(\mathcal{D}_d)$, where $H_p^*(\mathcal{D}_d)$ is described as in (b) above, and let S be the surface of a body B obtained by rotating the function $\phi(x)$ about the x-axis. The optimal rate of convergence of an approximation (1.3) to each component of the scattered electromagnetic field due to B obtained by solving the integral equation problem in [12] is $O(e^{-cn^{1/2}})$, where c is subject to (1.9). The method used in [12] converges at the $O(\exp\{-(2^{-1/2}\, a-\varepsilon)n^{1/2}\})$ rate.

(l) Consider the solution of the linear problem

$$(1.23) \qquad \frac{d^2f}{dz^2} + p(z)\frac{df}{dz} + q(z)f = r(z) \ , \ f(-1) = f(1) = 0 \ ,$$

for f, given p, q and r are analytic in $\mathcal{D}_d$ as defined in (a). Assume furthermore, that we can determine a priori, the existence of a solution of (1.20), with the property that the functions $(1-z^2)f''(z)$, $(1-z^2)p(z)f'(z)$, $(1-z^2)q(z)f(z)$ and $(1-z^2)r(z)$ are in $H_p(\mathcal{D}_d)$, as defined in (a) above. Then the optimal rate of convergence of an approximate solution (1.3) to f is $O(e^{-cn^{1/2}})$, where c is subject to (1.9). The method in [20] produces a solution of the form (1.10) which converges at the $O(\exp\{-(2^{-1/2}a-\varepsilon)n^{1/2}\})$ rate, where $n = 2N+1$.

(m) We can also state optimal rates of convergence similar to those in (l) above for solutions of differential equations over $(0,\infty)$ and over $(-\infty,\infty)$; moreover, methods are derived in [20] which converge at the $O(\exp\{-(2^{-1/2}a-\varepsilon)n^{1/2}\})$ rate. The methods apply also to the case of nonlinear equations, and to the solution of partial differential equations. In the latter case, the best possible rate of convergence in s dimensions* is $O(e^{-cn^{1/(2s)}})$; this functional form of convergence is achieved in [20] for the case of the approximate solution of partial differential equation problems in two dimensions. Similarly, the best possible functional form

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*Consider the approximation of $u(x,y)$ on the square $S = [-1,1]\times[-1,1]$ by by n^2 basis functions of the form $\{\phi_{n,j}(x)\ \psi_{n,\ell}(y)\}_{j,\ell=1}^{n}$. Let u considered as an function of x (rsp. y) satisfy the conditions of Thm 1.2 for each fixed y (rsp. x) on $[-1,1]$. Then by Thm. 1.2 there exists a $u(\cdot,y)$ such that $v \in H_p(U)$, where $v(x,y) = (1-x^2)^{-1}\ u(x,y)$, and such that

$$\inf_{w_j,v_\ell,x_k,y_\ell}\ \sup_{(x,y)\in S} \left| u(x,y) - \sum_{j=1}^{n} w_j\Big\{\sum_{\ell=1}^{n} v_\ell u(x_j,y_\ell)\psi_{m,\ell}(y)\Big\} \circ \phi_{n,j}(x)\right|$$
$$\geq e^{-cn^{1/2}}\,||u(\cdot,y)||_p$$

where c is subject to (1.9), in which a is defined by (1.6), with $d = \pi/2$. On the other hand, a Cartesian product formula of the type (1.10) using n^2 basis functions may be used to approximate u on S to within an $\exp\{-(2^{-1/2}a-\varepsilon)n^{1/2}\}$ error. That is using n basis functions, we achieve a rate of convergence of $e^{-c'n^{1/4}}$. Similarly, it is possible to achieve an optimal convergence rate $e^{-c|n|/(2s)}$ using n basis functions in s dimensions.

for the rate of convergence is achieved in [14] for the case of the approximate solution of integral equations, in the space of functions considered in [14].

In light of the proof of the upper bound in Thm. 1.2 (see also [4,5]), the present paper shows that when approximating functions that are in $H_p(U)$ over $(-1,1)$ we can do no better with regards to achieving the best $e^{-cn^{1/2}}$ rate, than one can do using rational functions as bases, or for that matter, the functions used in (1.10).

The present paper is also related, in the sense of optimal quadratures, with the work of Sobolev [16] and other Russian mathematicians (see e.g. [24]) who have verified optimality of certain quadrature rules with respect to Sobolev norms. Such norms involve the integral of a power of one or more derivatives, and consequently, using an n-point method, the optimal rate of convergence is $O(n^{-c})$. Sobolev was primarily interested in obtaining optimal methods of solving elliptic partial differential equations via integral equation methods. In this regard, we believe that the rules of e.g. [19] have some advantage over Sobolev-type methods, since in applications, the solution u of an elliptic PDE is analytic, a.e., and while it is usually not possible to determine the exact nature of the singularities of u, it is possible to determine a priori the points, or curves or surfaces on which the singularities of u occur. That is, it is then possible to solve for u via an $e^{-cn^{1/(2s)}}$ method in s dimensions.

2. The Upper Bound in Theorem 1.1

We find from [19, Eq. (7.4)], that if $f \in H_p(U)$, then for any $h > 0$,

$$(2.1)\qquad \left| \int_{-1}^{1} f(x)dx - h \sum_{j=-\infty}^{\infty} \frac{2e^{jh}}{(1+e^{jh})^2} f\left(\frac{e^{jh}-1}{e^{jh}+1}\right) \right|$$

$$\leq \frac{e^{-\pi^2/(2h)}}{2 \sinh\left(\frac{\pi^2}{2h}\right)} \lim_{r\to 1^-} \int_0^{2\pi} |f(re^{i\theta})| d\theta$$

$$\leq 2\pi ||f||_p \frac{e^{-\pi^2/h}}{1-e^{-\pi^2/h}} \quad .$$

Lemma 2.1: Let $1 < p \leq \infty$, and let $f \in H_p(U)$. Then there exists a constant C_p such that for all $x \in (-1,1)$,

$$(2.2)\qquad |f(x)| \leq C_p ||f||_p (1 - x^2)^{-1/p}$$

Proof: The Poisson formula yields

$$f(x) = \lim_{r\to 1^-} \frac{(1 - x^2)}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})\, d\theta}{1 - 2x \cos\theta + x^2}$$

for any $x \in (-1,1)$, and therefore

$$(2.4)\qquad |f(x)| \leq (1 - x^2) ||f||_p \; G_q(x)$$

where by Eq. 15.3.3 of [1]

$$(2.5)\qquad G_q^q(x) = 2(1 + x)^{-q} F(\tfrac{1}{2}, q; 1; \frac{4x}{(1+x)^2})$$

$$= 2(1 - x)^{1-2q} F(\tfrac{1}{2}, 1 - q; 1; \frac{4x}{(1+x)^2})$$

$$\sim 2(1-x)^{1-2q}\,\frac{\Gamma(q-\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(q)}$$

as $x \to 1$.

The inequality (2.2) now follows from (2.4) and (2.5).

Now, by (2.1) and (2.2), we have

$$\left| \int_{-1}^{1} f(x)dx - h \sum_{j=-N}^{N} \frac{2e^{jh}}{(1+e^{jh})^2} f\left(\frac{e^{jh}-1}{e^{jh}+1}\right) \right|$$

$$\leq \{C_1' e^{-\pi^2/h} + 2C_p h \sum_{j=N+1}^{\infty} \frac{2e^{jh}}{(1+e^{jh})^2} [1 - (\frac{e^{jh}-1}{e^{jh}+1})^2]^{-1/p}\} \cdot ||f||_p \tag{2.6}$$

$$\leq \{C_1' e^{-\pi^2/h} + C_2^1 e^{-Nh/q}\} ||f||_p$$

$$\leq C e^{-\{\pi/(2q)^{\frac{1}{2}}\} n^{\frac{1}{2}}}$$

where $n = 2N + 1$, and where we replaced h by $\pi q^{\frac{1}{2}}/N^{\frac{1}{2}}$.

3. The Lower Bound in Theorem 1.1

The function

$$f_t(z) = \left(\frac{1+z}{2}\right)^t, \qquad t \geq 0, \tag{3.1}$$

is in $H_p(U)$, and it has norm

$$||f_t||_p \leq ||f_t||_\infty = 1 . \tag{3.2}$$

We also have

$$\begin{cases} \displaystyle\int_{-1}^{1} f_t(z)dz = \frac{2}{1+t} \\ \displaystyle\sum_{j=1}^{n} w_j^* f_t(z_j^*) = \sum_{j=1}^{n} w_j^* e^{-a_j^* t} \end{cases} \tag{3.3}$$

where the second of these is a quadrature formula for approximating $\int_{-1}^{1} f(z)dz$ for any $f \in H_p(U)$, and where

$$\text{Re } a_j^* = - \text{ Re log } \left(\frac{1+z_j^*}{2}\right) > 0 , \tag{3.4}$$

since it is readily seen that if $|z_j^*| = 1$ for some j, then $\sum_{j=1}^n w_j^* f(z_j^*)$ will be infinite for some functions $f \in H_p(U)$.

Hence we now have, in view of (1.2), that

$$\begin{aligned} \sigma_n \geq \gamma_n &= \inf_{w_j^*, a_j^*} \sup_{t>0} \left| \frac{2}{1+t} - \sum_{j=1}^{n} w_j^* e^{-a_j^* t} \right| \\ &= \sup_{t>0} \left| \frac{2}{1+t} - \sum_{j=1}^{n} w_j e^{-a_j t} \right| \end{aligned} \tag{3.5}$$

By taking the Laplace transform of each side of (3.5) we find that

(3.6)
$$\gamma_n \geq \sup_{s>0} \left| \phi(s) - \sum_{j=1}^{n} \frac{w_j s}{s + a_j} \right|$$

where

(3.7)
$$\phi(s) = \int_0^\infty \frac{2se^{-st}}{1 + t}\, dt \ .$$

Lemma 3.1: Let ϕ be given by (3.7). Then

(3.8)
$$\phi(s) = 2se^s \log \frac{1}{s} + g(s)$$

where g is an entire function. Moreover ϕ is uniformly bounded on $[0,\infty]$.

Proof: The representation and everywhere analyticity of g follows by inspection of the identity

(3.9)
$$\phi(s) = 2se^s \log \frac{1}{s} + 2se^s \log(s + a)$$
$$+ \int_0^a \frac{2s[e^{-u} - e^s]}{s + u}\, du + 2s \int_0^\infty \frac{e^{-u-a}}{s + u + a}$$

which is valid for all $a > 0$. The boundedness of ϕ on R is a consequence of

$$\phi(s) = \int_0^\infty \frac{2se^{-u}}{s + u}\, du \leq 2 \int_0^\infty e^{-u} du = 2 \ .$$

We now note that

(3.10)
$$\gamma_n \geq \sup_{s\in\mathbb{R}} \left| \phi(s^2) - \sum_{j=1}^{n} \frac{w_j s^2}{s^2 + a_j} \right|$$

$$\geq \tau_n = C_a \left(\int_R \left| \frac{\phi(s^2)}{s^2 + a} - \sum_{j=1}^{n} \frac{w_j s^2}{(s^2 + a_j)(s^2 + a)} \right|^2 ds \right)^{1/2}$$

where $a > 0$, and where

(3.11)
$$C_a = \left(\int_R \frac{ds}{(s^2 + a)^2}\right)^{-1/2} = \left(\frac{2}{\pi}\right)^{1/2} a^{3/4} .$$

Setting

(3.12)
$$(Hf)(x) = -\frac{P.V.}{\pi} \int_R \frac{f(t)}{t - x} \, dt$$

for any $f \in L^2(R)$, where P.V. denotes the principal value, it follows that

(3.13)
$$||Hf||_2 = ||f||_2 .$$

Let us define Φ by

(3.14)
$$\Phi(x) = H\left\{\frac{\phi(s^2)}{s^2 + a}\right\} (x)$$

and let us note that if a and b are positive numbers, then

(3.15)
$$H\left\{\frac{s^2}{(s^2 + a)(s^2 + b)}\right\}(x) = \frac{x}{a^{1/2} + b^{1/2}} \frac{x^2 - a^{1/2}b^{1/2}}{(x^2 + a)(x^2 + b)} ,$$

where this identity may be established via the use of Fourier transforms. Let us define polynomials p and q of degree $\leq n$ by

$$\frac{p(x^2)}{q(x^2)} = \sum_{j=1}^{n} \frac{w_j}{a^{\frac{1}{2}} + a_j^{\frac{1}{2}}} \frac{x^2 - a^{\frac{1}{2}} a_j^{\frac{1}{2}}}{x^2 + a_j^{\frac{1}{2}}} \tag{3.16}$$

It then follows from (3.10), (3.13), (3.14), (3.15) and (3.16), that

$$\tau_n = C_a \left(\int_R \left| \phi(x) - \frac{x}{x^2 + a} \frac{p(x^2)}{q(x^2)} \right|^2 dx \right)^{\frac{1}{2}} \tag{3.17}$$

<u>Lemma 3.2</u>: Let E_ρ denote the ellipse in the complex plane, having foci at ± 1 and having sum of semi-axes equal to $\rho > 1$. In (3.14) and (3.16), let $a = 2 \sinh(1) = e - e^{-1}$. Then on $[-1,1]$

$$\phi(x) = \frac{\pi}{2} \frac{x^2 e^{x^2}}{a + x^2} \operatorname{sgn} x + \psi(x) \tag{3.18}$$

where $\psi(-x) = -\psi(x)$, and where ψ is the restriction to $[-1,1]$ of a function which is analytic and bounded in E_e.

<u>Proof</u>: Let us set $w(s) = \phi(s^2)/(s^2 + a)$. The functions w and Φ are conjugate harmonic functions and therefore Φ is analytic wherever w is analytic. The singular behavior of Φ at $x = 0$ is determined by considering the first of the two integrals

$$\Phi(x) = -\frac{\text{P.V.}}{\pi} \int_{-2}^{2} \frac{w(s)}{s - x} ds - \frac{1}{\pi} \int_{|s|>2} \frac{w(s)}{s - x} ds . \tag{3.19}$$

The second integral on the right is clearly analytic at $x = 0$; on the other hand

$$-\frac{\text{P.V.}}{\pi} \int_{-2}^{2} \frac{s^{2n} \log(1/|s|)}{s - x} ds = \frac{\pi}{2} x^{2n} \operatorname{sgn} x + g_n(x) \tag{3.20}$$

where g_n is analytic at $x = 0$. The representation (3.18) thus follows.

The analyticity and boundedness of ψ in E_e is a consequence of the analyticity of $s^2 e^{s^2}/(a+s^2)$ in $E_{(2e)}$; the relation $\Phi(-x) = -\Phi(x)$ follows from (3.14), since $w(-s) = w(s)$.

Let $m = [n]^{3/4}$ and let u_m and v_{m-1} be polynomials such that

$$
(3.21) \quad \begin{cases} \varepsilon_1 = \inf_{u \in P_m} \sup_{x \in [-1,1]} \left| 2e^{x^2} - u(x^2) \right| = \sup_{x\ [-1,1]} \left| e^{x^2} - u_m(x^2) \right| \\ \varepsilon_2 = \inf_{v \in P_{m-1}} \sup_{x \in [-1,1]} \left| \frac{\psi(x)}{x} - v(x^2) \right| = \sup_{x\ [-1,1]} \left| \frac{\psi(x)}{x} - v_{m-1}(x^2) \right| \end{cases}
$$

where P_m denotes the family of all polynomials of degree $\leq m$. Then [22]

$$
(3.22) \qquad |\varepsilon_1| + |\varepsilon_2| \leq (C/C_a) e^{-m}
$$

where C_a is as in (3.11), and where C is independent of m .

Now, from (3.17), (3.18) and (3.21),

$$
\begin{aligned}
(3.23) \qquad \tau_n &\geq C_a \left(\int_{-1}^{1} \left| \Phi(x) - \frac{x}{x^2 + a} \frac{p(x^2)}{q(x^2)} \right|^2 dx \right)^{\frac{1}{2}} \\
&\geq C_a \left(\int_{-1}^{1} \left| \frac{x^2 u_m(x^2)}{a + x^2} \operatorname{sgn} x + x\, v_{m-1}(x^2) - \frac{x}{x^2 + a} \frac{p(x^2)}{q(x^2)} \right|^2 dx \right)^{\frac{1}{2}} \\
&\quad - Ce^{-m} \\
&\geq \frac{2C_a}{1 + a} \left(\int_{0}^{1} \left| x^2 u_m(x^2) + x(a + x^2) v_{m-1}(x^2) - x \frac{p(x^2)}{q(x^2)} \right|^2 dx \right)^{\frac{1}{2}} \\
&\quad - Ce^{-m} ,
\end{aligned}
$$

the last inequality being a consequence of $a + x^2 \leq a + 1$ on $[0,1]$.

Upon setting

$$(3.24)\qquad \begin{cases} P(x^2) = p(x^2) - (a + x^2)v_{m-1}(x^2)q(x^2) \\ Q(x^2) = u_m(x^2)q(x^2) \end{cases}$$

we have $P,Q \in \mathbb{P}_{m+n}$, and so (3.23) yields

$$(3.25)\qquad \begin{aligned} \tau_n &\geq \frac{2C_a}{1+a}\left[\int_0^1 x^4\left|\frac{P(x^2) - xQ(x^2)}{x\,q(x^2)}\right|^2 dx\right]^{\frac{1}{2}} - Ce^{-m} \\ &= \frac{2C_a}{1+a}\left(\int_0^1 x^4 u_m^2(x^2)\left|\frac{P(x^2) - xQ(x^2)}{xQ(x^2)}\right|^2 dx\right)^{\frac{1}{2}} - Ce^{-m}. \end{aligned}$$

Next, (3.21) yields $u_m(x^2) > 1$ on $[0,1]$; therefore

$$(3.26)\qquad \tau_n \geq \frac{2C_a}{1+a}\left(\int_0^1 x^4\left|\frac{P_N(x^2) - x\,Q_N(x^2)}{x\,Q_N(x^2)}\right|^2 dx\right)^{\frac{1}{2}} - Ce^{-m} .$$

In (3.26) and in what follows

$$(3.27)\qquad \begin{cases} N = n + m \\ \gamma_N = \inf\limits_{\substack{P,Q\in\mathbb{P}_N \\ Q(\frac{1}{2})>0}} \displaystyle\int_0^1 x^4\left|\frac{P(x^2) - x\,Q(x^2)}{x\,Q(x^2)}\right|^2 dx \\ \quad = \displaystyle\int_0^1 x^4\left|\frac{P_N(x^2) - xQ_N(x^2)}{x\,Q_N(x^2)}\right|^2 dx . \end{cases}$$

<u>Lemma 3.3</u>: $Q_N(x^2) > 0$ on $(0,1]$.

<u>Proof</u>: We may assume without loss of generality that $P_N - x\,Q_N$ and $x\,Q_N$ have no common zeros on $[0,1]$. However, in this case we cannot

have $Q_N(\xi^2) = 0$ for some $\xi \in (0,1]$, since γ_N would then be infinite. In view of (3.27), we must have $Q_N > 0$ on $(0,1]$.

<u>Lemma 3.4</u>: Let S be the small subset of $[0,1]$ on which $P_N(x^2) < 0$. Then, with $I = [0,1]$

$$\gamma_N \geq \inf_{p \in \mathbb{P}_{2N+1}} \int_{I-S} x^4 \left|\frac{p(x)}{p(-x)}\right|^2 dx + \int_S x^4 dx \tag{3.28}$$

<u>Proof</u>: From (3.27), we have, since $P_N(x^2) \geq 0$ on $I - S$, and $P_N(x^2) < 0$ on S, that

$$\left|\frac{P_N(x^2) - x\, Q_N(x^2)}{x\, Q_N(x^2)}\right| > 1 \quad \text{on} \quad S \tag{3.29}$$

and

$$\left|\frac{P_N(x^2) - x\, Q_N(x^2)}{x\, Q_N(x^2)}\right| \geq \left|\frac{P_N(x^2) - x\, Q_N(x^2)}{P_N(x^2) + x\, Q_N(x^2)}\right| \quad \text{on} \quad I - S\ , \tag{3.30}$$

and therefore

$$\gamma_N \geq \int_{I-S} x^4 \left|\frac{P_N(x^2) - x\, Q_N(x^2)}{P_N(x^2) + x\, Q_N(x^2)}\right|^2 dx + \int_S x^4 dx \tag{3.31}$$

from which (3.28) follows.

<u>Lemma 3.5</u>: Let $q \in \mathbb{P}_{2N+1}$ be defined by

$$\int_{I-S} x^4 \left|\frac{q(x)}{q(-x)}\right|^2 dx = \inf_{p \in \mathbb{P}_{2N+1}} \int_{I-S} x^4 \left|\frac{p(x)}{p(-x)}\right|^2 dx\ . \tag{3.32}$$

Then all zeros of q are on $[0,1]$.

<u>Proof</u>: Clearly $I - S$ must have positive measure, for otherwise (3.31)

would imply that $\gamma_N \geq 1/5$, contradicting (2.6) . Let ξ be a zero of q which is not on $[0,1]$. Then set

(i) $$q_1(x) = \frac{x+\xi}{x-\xi}\,q(x) \qquad \text{if } \xi \in [-1,0)$$

(ii) $$q_1(x) = \frac{1+x\xi}{x-\xi}\,q(x) \qquad \text{if } \xi < -1$$

(iii) $$q_1(x) = \frac{1-x\xi}{x-\xi}\,q(x) \qquad \text{if } \xi > 1$$

(iv) $$q_1(x) = \frac{x-u}{x-u-iv}\,q(x) \qquad \text{if } \xi = u+iv,\ v \neq 0 .$$

In all* of the these cases

$$\left|\frac{q_1(x)}{q_1(-x)}\right| < \left|\frac{q(x)}{q(-x)}\right| \quad \text{a.e. on } [0,1] .$$

This however contradicts the definition of q, since $I - S$ has positive measure.

<u>Lemma 3.6</u>: Let γ_N be defined as in (3.27). Then

(3.33) $$\gamma_N \geq \inf_{p\in\mathcal{P}_{2N+1}} \int_0^1 x^4 \left|\frac{p(x)}{p(-x)}\right|^2 dx \geq$$

$$2\pi 5^{-\frac{1}{4}}(N+\tfrac{1}{2})^{\frac{1}{4}} \exp[-2\pi 5^{\frac{1}{4}}(N+\tfrac{1}{2})^{\frac{1}{2}}]$$

<u>Proof</u>: By Lemma 3.5, all of the $2N+1$ zeros of q defined in (3.32) are on $[0,1]$ and therefore

---

*If in (iv) we have $|u| > 1$, then $q_1(x)/q_1(-x)$ can be further reduced in magnitude a.e. on $[0,1]$ via the use of (ii) or (iii).

(3.34) $$\left|\frac{q(x)}{q(-x)}\right| = \prod_{i=1}^{2N+1} \left|\frac{x-\xi_i}{x+\xi_i}\right| \le 1 \quad \text{a.e. on} \quad [0,1] .$$

Lemma (3.4) now yields

$$\gamma_N \ge \int_{I-S} x^4 \left|\frac{q(x)}{q(-x)}\right|^2 dx + \int_S x^4 dx$$

$$\ge \int_0^1 x^4 \left|\frac{q(x)}{q(-x)}\right|^2 dx$$

from which the first inequality on the right-hand side of (3.33) follows. The second inequality on the right-hand side of (3.33) is due to Rahman and Schmeisser [13]; this was established via a procedure resembling that of Newman [11].

Let us now complete the proof of the lower bound in (2.5). By (3.5), (3.10), (3.26), (3.27), (3.33) and $m = [n^{3/4}]$ we have

(3.36) $$\sigma_n \ge \gamma_n \ge \tau_n \ge \frac{2C_a}{1+a}\,[2\pi 5^{-1/2}(n + n^{3/4} + 1/2)^{1/2}]^{1/2} \cdot$$
$$\cdot \exp[-5^{1/2}\pi(n + n^{3/4} + 1/2)^{1/2}] - C^1 e^{-n^{3/4}}$$

Hence given any $\varepsilon > 0$ there exists an $n(\varepsilon) > 0$, such that

(3.37) $$\sigma_n \ge \exp[-(5^{1/2}\pi + \varepsilon)n^{1/2}]$$

for all $n \ge n(\varepsilon)$.

This completes the proof.

4. The Upper Bound in Theorem 1.2

The function ρ_N (see [2]) defined by

$$(4.1)\qquad \rho_N(z) = \prod_{j=1}^{N} \left(\frac{a_j^2 - x^2}{1 - a_j^2 x^2}\right)$$

$$a_j = \sqrt{k}\,\mathrm{sn}[\frac{2j-1}{2N} K ; k] , \qquad 0 < k < 1$$

has the following properties*:

$$(4.2)\qquad |\rho_N(x)| \le (k_{2N})^{\frac{1}{2}} \le 2\lambda^{N/2} \qquad \text{if } -k^{\frac{1}{2}} \le x \le k^{\frac{1}{2}}$$

$$\lambda = \exp(-\pi K'/K)$$

$$(4.3)\qquad |\rho_N(z)| \le 1 \qquad \text{if } z \in U;\ |\rho_N(z)| = 1 \text{ if } |z| = 1 .$$

For $x \in (-1,1)$, we consider the expression

$$(4.4)\qquad \begin{aligned} \varepsilon_N(x) &= \lim_{r\to 1^-} \frac{1}{2\pi i} \int_{|z|=1} \frac{\rho_N(x)}{\rho_N(z)} \frac{(1 - x^2)f(rz)dz}{(z - x)(1 - xz)} \\ &= f(x) - r_N(x) \end{aligned}$$

where r_N is a rational function of degree $\le 2N + 1$.

Furthermore, in view of (4.2) and (4.3), we have

$$(4.5)\qquad |\varepsilon_N(x)| \le \begin{cases} 2\lambda^{N/2}||f||_p^*(1 - x^2)G_q(x) & \text{if } -k^{\frac{1}{2}} \le x \le k^{\frac{1}{2}} \\ ||f||_p^*(1 - x^2)G_q(x) & \text{if } x \in (-1,1),\ |x| \ge k^{\frac{1}{2}} \end{cases} ,$$

where

*In [15 p. 326] one finds the relationships

$$(k_{4m})^{\frac{1}{2}} = \eta_m^m = \frac{2\lambda^m + 2\lambda^{9m} + 2\lambda^{25m} + \dots}{1 + 2\lambda^{4m} + 2\lambda^{16m} + \dots} , \text{ yielding (4.2).}$$

$$(4.6)\qquad ||f||_p^* = \lim_{r\to 1^-} \left(\frac{1}{2\pi} \int_{|z|=1} \frac{|f(rz)|^p |dz|}{|1 - r^2 z^2|^p} \right)^{1/p}$$

$$(4.7)\qquad G_q(x) = \lim_{r\to 1^-} \left(\frac{1}{2\pi} \int_{|z|=1} \frac{|1 - r^2 z^2|^q |dz|}{|1 - 2x\cos\theta + x^2|^q} \right)^{1/q} \; ; \; z = e^{i\theta},$$

and where $1/p + 1/q = 1$. We thus find that

$$(4.8)\qquad \begin{aligned} G_q(x) &= \frac{2}{1+x^2} \left(\frac{\Gamma\left(\frac{q}{2} + \frac{1}{2}\right)}{\pi^{1/2}\Gamma(\frac{q}{2} + 1)} \right)^{1/q} [F(\tfrac{q}{2}, \tfrac{q}{2} + \tfrac{1}{2} ; \tfrac{q}{2} + 1 ; \tau^2)]^{1/q} \\ \tau &= \frac{4x}{1+x^2} \end{aligned}$$

By formula 15.3.3. of [1] we may rewrite (4.8) as

$$G_q(x) = 2(1 - x^2)^{1/q-1}(1 + x^2)^{-1/q} \left| \frac{\Gamma(\frac{q}{2} + \frac{1}{2})}{\pi^{1/2}\Gamma(\frac{q}{2} + 1)} F(\tfrac{1}{2}, 1, \tfrac{q}{2} + 1 ; \tau^2) \right|^{1/q}$$

$$(4.9)\qquad \leq \begin{cases} 2(1 - x^2)^{1/q-1} \left(\dfrac{2\Gamma(\frac{q}{2} + \frac{1}{2})}{\pi^{1/2}(q - 1)\Gamma(q/2)} \right)^{1/q} & \text{if } q > 1 \\ \leq \dfrac{1}{\pi} \log \left(\dfrac{2\sqrt{2}}{1 - x^2}\right) & \text{if } q = 1 \end{cases}$$

Substituting (4.9) into (4.5), we get for $q > 1$:

$$|\epsilon_N(x)| \leq 4 \left(\frac{2\Gamma(\frac{q}{2} + \frac{1}{2})}{\pi^{1/2}(q - 1)\Gamma(q/2)} \right)^{1/q} (1 - x^2)^{1/q} \lambda^{N/2} ||f||_p^*$$

$$\text{if} \quad -k^{1/2} \leq x \leq k^{1/2}$$

$$(4.10) \qquad 4\left(\frac{2\Gamma(\frac{q}{2}+\frac{1}{2})}{\pi^{\frac{1}{2}}(q-1)\Gamma(q/2)}\right)^{1/q} (1-x^2)^{1/q}||f||_p^*$$

$$\text{if } x \in (-1,1)\ , \quad |x| \geq k^{\frac{1}{2}}\ .$$

while if $q = 1$

$$|\epsilon_N(x)| \leq \frac{1}{\pi}(1-x^2)\log\left(\frac{2\sqrt{2}}{1-x^2}\right)\lambda^{N/2}||f||_\infty^*$$

$$\text{if } -k^{\frac{1}{2}} \leq x \leq k^{\frac{1}{2}}, \text{ and}$$

$$(4.11) \qquad |\epsilon_N(x)| \leq \frac{1}{\pi}(1-x^2)\log\left(\frac{2\sqrt{2}}{1-x^2}\right)||f||_\infty^*$$

$$\text{if } x \in (-1,1) \text{ and } |x| \geq k^{\frac{1}{2}}\ .$$

Upon noting that [17, Eq. (2.23)]

$$(4.12) \qquad 1 - k \leq 8\exp\left[\frac{-\pi^2}{\log 1/\lambda}\right]$$

we find that

$$(4.13) \quad |\epsilon_N(x)| \leq \begin{cases} C_q||f||_p^* \exp\left[-\frac{N}{2}\log\frac{1}{\lambda}\right], & 0 \leq |x| \leq k^{\frac{1}{2}}, \\ C_q||f||_p^* \exp\left[\frac{-\pi^2 q}{\log\frac{1}{\lambda}}\right], & k^{\frac{1}{2}} \leq |x| \leq 1\ . \end{cases}$$

if $q > 1$, and where C_q depends only on q, and

$$(4.14) \quad |\epsilon_N(x)| \leq \begin{cases} \frac{||f||_p^*}{\pi}\log(2\sqrt{2})\exp\left[-\frac{N}{2}\log\frac{1}{\lambda}\right] & \text{if } 0 \leq |x| \leq k^{\frac{1}{2}}, \\ \frac{8\pi}{\log\frac{1}{\lambda}}\exp\left[-\frac{\pi^2}{\log\frac{1}{\lambda}}\right] & \text{if } k^{\frac{1}{2}} \leq |x| \leq 1 \end{cases}$$

if $q = 1$.

By taking $\log(1/\lambda) = 2^{\frac{1}{2}}\pi/N^{\frac{1}{2}}$, we find, in all cases, that

$$|\varepsilon_N(x)| \leq C_p^1 ||f||_p^* (N/2)^{\frac{1}{2}} \exp\left[\frac{-\pi}{(2q)^{\frac{1}{2}}} N^{\frac{1}{2}}\right], \tag{4.15}$$

which, upon replacing N by $(n-1)/2$, yields the upper bound in Theorem 1.2 .

We remark that the same upper bound is obtainable using approximations derived in Theorem 7.1 of [10]. Since the approximation technique of [10] does not involve rational approximations, this suggests strongly that the constants in the upper bounds in Theorems 1.1 and 1.2 cannot be improved.

5. The Lower Bound in Theorem 1.2

Since $g \in H_p(U)$, where $g(z) = f(z)/(1 - z^2)$, Theorem 1.1 implies that

$$(5.1) \qquad \exp[-(5^{\frac{1}{2}}\pi + \varepsilon)n^{\frac{1}{2}}]$$

$$\leq \sup_{f \in H_p^*(U), \|f\|_p^* = 1} \left| \int_{-1}^{1} \left[\frac{f(x)}{1 - x^2} - \frac{1}{1 - x^2} T_n(f)(x) \right] dx \right| .$$

By Lemma 2.1,

$$(5.2) \qquad \left| \frac{f(x)}{1 - x^2} \right| \leq C_p (1 - x^2)^{-1/p} , \quad -1 < x < 1$$

while by (1.4) and Lemma 2.1,

$$(5.3) \qquad \left| \frac{T_n(f)(x)}{1 - x^2} \right| \leq C\ C_p (1 - x^2)^{-1/p} .$$

Thus (5.1) implies that

$$(5.4) \qquad \exp[-(5^{\frac{1}{2}}\pi + \varepsilon)^{\frac{1}{2}} n^{\frac{1}{2}}]$$

$$\leq \varepsilon_{n,o} \int_{-1+\delta}^{1-\delta} \frac{dx}{1 - x^2} + 2\ C_p (1 + C) \int_{|x| \geq 1-\delta} (1 - x^2)^{-1/p} dx$$

where, subject to (1.4),

$$(5.5) \qquad \varepsilon_{n,o} = \inf_{T_n} \sup_{f \in H_p^*(U), \|f\|_p^* = 1} \sup_{-1<x<1} |f(x) - T_n(f)(x)|$$

that is, by taking $\delta = 2e^{-n}$, we have

$$\exp[-(5^{\frac{1}{2}}\pi + \varepsilon)n^{\frac{1}{2}}] \le \varepsilon_{n,o}n + 2q\, C_p(1 + C)e^{-n/q} \tag{5.6}$$

which yields the lower bound in Theorem 1.2 .

References

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